Dynamics as Shadow of Phase Space Geometry

J. R. Klauder[†] and P. Maraner[‡]

† *Departments of Physics and Mathematics,*
University of Florida, Gainesville, FL 32611, USA

‡ *Dipartimento di Fisica, Università di Parma,*
and INFN, Gruppo collegato di Parma,
Viale delle Scienze, 43100 Parma, Italy

Abstract

Starting with the generally well accepted opinion that quantizing an arbitrary Hamiltonian system involves picking out some additional structure on the classical phase space (the *shadow* of quantum mechanics in the classical theory), we describe classical as well as quantum dynamics as a purely geometrical effect by introducing a *phase space metric structure*. This produces an $\mathcal{O}(\hbar)$ modification of the classical equations of motion reducing at the same time the quantization of an arbitrary Hamiltonian system to standard procedures. Our analysis is carried out in analogy with the adiabatic motion of a charged particle in a curved background (the additional metric structure) under the influence of a universal magnetic field (the classical symplectic structure). This allows one to picture dynamics in an unusual way, and reveals a dynamical mechanism that produces the selection of the right set of physical quantum states.

Dynamics as Geometry

Paolo Maraner
Università di Parma
Dipartimento Fisico
Viale delle Scienze
43100 Parma
Italy

Telephone: +39-521-905281
Fax   : +39-521-905223
E-Mail : *maraner@parma.infn.it*

1   Introduction

The search for a *geometrical* description of the laws of nature is part of an important tradition in modern physics, and such techniques allow one to gain a coordinate-free viewpoint of the formal structures as well as the global features of a theory. Guided by this spirit, we would like to show how classical (Hamiltonian) as well as quantum dynamics may be formulated as the *adiabatic limit* of a *fully-geometrical phase-space-theory* constructed with the help of a metric and the standard symplectic [1, 2] phase space structures (see Eq.8 and Eqs.25,26 below). Our work is mainly motivated by the attempt to overcome various difficulties concerning the construction of a coordinate-free quantization procedure, a context in which the introduction of subsidiary geometrical phase space structures seems to be unavoidable.

As is well known, the standard way to look at quantization proceeds from Dirac's observation [3] that the necessity of interpreting every quantum phenomenon with classical expressions forces the formal structure of a quantum theory to be isomorphic to the one of the corresponding classical theory (correspondence principle). Therefore, quantization may be regarded as the attempt of building a bridge between the formal structures of classical and quantum mechanics, that is to say, to find a correspondence between classical and quantum *states*, *observables (kinematics)* and *evolution equations (dynamics)*. The conclusion emerging from many attempts at building a geometrical quantization procedure is that it is impossible to have a one-to-one correspondence between the algebra of classical and quantum observables without making the Hilbert space of the corresponding quantum system too large. Moreover the necessary selection of a subalgebra for which the correspondence holds may be regarded as picking out some *additional structure* on the classical phase space M. *"This* [additional structure] *can be thought as the* **shadow** *of quantum mechanics in the classical system and the element of choice in this selection is the (...) point at which we come across an ambiguity in passing from the classical to the quantum domain"* (N. Woodhouse [4], emphasis added).

The nature of the additional structure is today a matter of discussion.

3

The first concerted effort to overcome the difficulty goes back to the first Geometric Quantization papers [5, 6, 7]. It consists in picking out a *real* or a *complex polarization* on the phase space M—when there is one—and asserting that the physical states of the quantum system should in some sense preserve the polarization (see e.g. [4]). This prescription emerges from the analysis of a wide class of examples (with a high degree of symmetry). It gives correct physical answers for highly symmetrical systems [8] but appears problematic as soon as the dynamics of systems with less symmetry or no symmetry at all is considered. There is no longer any guarantee that the evolution of the system respects the polarization, and physical states may evolve into non-physical ones.

Additionally, physical insight into the problem may be gained by looking at the phase space path integral expression of the propagator

$$K(q'', t''; q', t') \stackrel{?}{=} \int e^{\frac{i}{\hbar} \int [p_\mu \dot{q}^\mu - h(q,p)] dt} \, \mathcal{D}q \mathcal{D}p. \tag{1}$$

This formal integral involves only the classical symplectic structure and superficially appears covariant under canonical transformations. It is on the other hand immediate that this canonical invariance must be broken. Otherwise the introduction of a suitable set of canonical variables would make the formal path integral expressions cooincide and hence make the spectra of distinct physical systems equal. This undesirable consequence is avoided when it is recognized that to be defined the formal integral needs *regularization* and that regularization—e.g. the commonly used lattice regularization—breaks canonical invariance. *It is the phase space structure producing the breakdown of canonical invariance in the conventioanl phase space path integral that can be identified with the* **shadow** *of quantum mechanics in the classical theory.* Being restricted to flat phase spaces—moreover to Cartesian coordinate frames—we cannot hope to gain real insight into the nature of the "additional structure" by considering lattice regularizations. In so doing one must confront the meaning of the formal expression $\mathcal{D}q\mathcal{D}p$, which on the surface appears to be a construct solely of the symplectic structure. Nevertheless, for a q to q propagator, as indicated in (1), a lattice formulation shows that in fact the symplectic structure is *not* involved; rather, there is

always one more p integration than q integration, and that these measures appear separately and they involve a configuration or momentum space *metric*, respectively. Extended in an invariant way to phase space it suggests that we seek a meaning of the formal expression of the functional measure $\mathcal{D}q\mathcal{D}p$ through the introduction of a *metric structure* on M. It is our opinion that this phase space metric structure represents the appropriate shadow of quantization and in some way should replace the notion of polarization inside the Geometric Quantization scheme.

A geometrical quantization procedure moving along these lines has in fact been proposed a few years ago by one of us, J. R. Klauder [9]. In that context the purpose of the (Riemannian) metric is to provide an adequate geometrical structure on phase space to support Brownian motion which is used to give a continuous-time regularization of the formal expression of the phase-space path integral

$$K(p'', q'', t''; p', q', t') \stackrel{\text{def}}{=} \lim_{\nu \to \infty} N_\nu \int e^{\frac{i}{\hbar} \int [p_\mu \dot{q}^\mu - h(q,p)] dt} \, d\mu_W^\nu \qquad (2)$$

where $d\mu_W^\nu$ denotes a Wiener measure on M constructed by means of the Riemannian metric g_{ij}, ν the Brownian diffusion constant and N_ν an appropriate, and well defined, ν-dependent normalization constant. In contrast to the situation depicted in (1) it should be noted that the Wiener measure on phase space with its pinning of paths at both the initial and final times leads automatically to an expression depending on p'', q'' as well as p', q'. For particular classes of metrics it is possible to demonstrate that the propagator (2) with $h(q,p) = 0$ behaves as the projector on the set of physical states [9, 10, 11, 12, 13, 14], while as soon as we consider $h(q,p) \neq 0$ we are sure that the states of the system evolve within the selected subspace. For highly symmetrical phase spaces, admitting a Kähler or a conformal Kähler structure, the kinematical scheme $[h(q,p) = 0]$ reproduces the same results as the introduction of a complex polarization. In some sense, therefore, the introduction of a Riemannian metric on the phase space M includes the idea of polarization, and, in addition, remains compatible with the introduction of dynamics.

Having motivated the phase space metric on mathematical grounds, we

would like to present a different and rather unconventional approach to the problem. We propose to regard the phase space metric structure as a concrete physical object—to be considered as fundamental as the symplectic structure and not as an artificial regulator (see also [15]). Then we suggest an $\hbar$-dependent modification of the laws of dynamics making the quantization problem into a rather trivial one. In our picture dynamics appears in an interesting way from the competition between the metric and symplectic phase space structures in close analogy with the guiding center motion of a charged particle on a plane in an *inhomogeneous* magnetic field [16, 17, 18, 19, 20, 21]. This analogy is actually very useful in picturing both the motion of the system and the dynamical mechanism that provides the selection of the right set of physical quantum states.

In section 2 we focus on classical dynamics. After briefly reviewing the coordinate-free formulation of classical dynamics—based on symplectic geometry—we introduce a metric structure on the phase space and we illustrate how Hamiltonian mechanics may be described as the adiabatic limit of a fully-geometrical phase-space-theory. We also discuss the analogy of our model with the motion of a charged particle on a manifold in an inhomogeneous magnetic field. As a concrete example the harmonic oscillator problem is worked out in some detail. The problem of quantization is faced in section 3. We start again by briefly reviewing the mathematical tools necessary for the construction of a coordinate-free quantization procedure, and, by considering the "magnetic analogy", we illustrate how this mathematical background is natural in our formulation. We then proceed to discuss the quantization of our 'free' phase space theory.

Throughout this paper we employ the convention that a sum over repeated indices is implied. Phase space coordinates are denoted in a compact manner, and they have the dimension of the square root of an action.

2 Hamiltonian Dynamics as 'Free' Dynamics

2.1 Kinematics and Symplectic Geometry

In the language of modern differential geometry the phase space of an n degree of freedom Hamiltonian system is described by a $2n$-dimensional manifold M equipped with a closed, nondegenerate two-form, the *symplectic form*[1] ω_{ij} [1, 2]. This geometrical structure, in fact, represents all that is necessary to take into account the kinematical properties of the system, the symplectic form being equivalent to the assignment of a Poisson bracket structure on the phase space. Introducing local coordinates $\xi = (\xi^i; i = 1, ..., 2n)$ on M the components of the symplectic two-form are interpreted as (minus) the Lagrange brackets between the phase space coordinates $[\![\xi^i, \xi^j]\!] = \omega_{ji}$, so that the fundamental Poisson brackets may be obtained as

$$\{\xi^i, \xi^j\} = \bar{\omega}^{ji}, \tag{3}$$

$\bar{\omega}^{ij}$ being the antisymmetric two-tensor defined in every coordinate system by the well-known relation between Lagrange and Poisson brackets, $\omega_{ik}\bar{\omega}^{jk} = \delta_i^j$. This completely characterizes the canonical structure of the system, that is the kinematics.

The description of dynamics, on the other hand, requires the specification of a smooth function on M, the *Hamiltonian* $h(\xi)$, an object which is not related to any geometrical feature of the phase space. Representing the symplectic two-form by means of the canonical one-form θ_i, $\omega_{ij} = \partial_i\theta_j - \partial_j\theta_i$, a very convenient way to assign dynamics is by means of Hamilton's variational principle

$$\delta \int \left(\theta_i\dot{\xi}^i - h(\xi)\right) dt = 0. \tag{4}$$

For a general phase space, θ_i may be defined only locally and up to the gradient of an arbitrary function of ξ, $\theta_i \to \theta_i + \partial_i G$, an arbitrariness which does not affect the results of the theory.

---

[1]Throughout this paper we shall denote forms and tensors by means of their local components in a given coordinate frame, e.g. $\omega = \omega_{ij}\, d\xi^i \wedge d\xi^j$.

Canonical Coordinates

This phase-space-covariant formulation of mechanics assumes a more familiar look once *canonical coordinates* are introduced. A theorem of Darboux asserts that it is possible to find local coordinates such that ω_{ij} as well as $\bar{\omega}^{ij}$ reduce to the standard form

$$\omega_{ij} = \bar{\omega}^{ij} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{5}$$

where I represents the n-dimensional identity matrix. Denoting phase space coordinates by $\xi = (q^1, ..., q^n, p_1, ...p_n)$, the fundamental Poisson brackets (3) assume the canonical form

$$\begin{aligned} \{q^\mu, q^\nu\} &= 0, \\ \{q^\mu, p_\nu\} &= \delta^\mu_\nu, \\ \{p_\mu, p_\nu\} &= 0, \end{aligned} \tag{6}$$

$\mu, \nu = 1, ..., n$. Up to the gradient of an arbitrary function of ξ, the canonical one-form may be chosen as $\theta_i = (p_1, ..., p_n, 0, ..., 0)$ so that (4) reduces to the standard expression

$$\delta \int \left(p_\mu \dot{q}^\mu - h(q, p) \right) dt = 0. \tag{7}$$

Darboux's coordinates are therefore to be identified with canonical coordinates. In the rest of this paper we suppose that the phase space M is parametrized by means of canonical coordinate frames. Nevertheless, in order to simplify the notation and to express our result in a phase-space-covariant manner, we continue to denote phase space coordinates by means of the single variable $\xi = (q^1, ..., q^n, p_1, ..., p_n)$.

2.2 Dynamics and Metric Geometry

We now come to the heart of our analysis. The global formulation of Hamiltonian mechanics makes it clear that whereas the kinematical properties of a

dynamical system are completely taken into account by a geometrical structure, the symplectic form ω_{ij}, dynamics is described by means of a non-geometrical object, the Hamiltonian $h(\xi)$. It is the purpose of this section to demonstrate that the dynamical properties of a Hamiltonian system may be understood as consequence of a second geometrical structure on the phase space, a *metric* g_{ij}. To be more precise we claim that

Introducing a metric $\mu g_{ij}(\xi)$ on the phase space M of a Hamiltonian system ($\mu^{1/2}$ being a parameter in which the scale of the phase space line-element ds is reabsorbed) in the limit of very small values of μ, the variational principle

$$\delta \int \left(\frac{1}{2}\mu g_{ij}\dot{\xi}^i\dot{\xi}^j + \theta_i\dot{\xi}^i \right) dt = 0 \tag{8}$$

produces the same dynamics as Hamilton's variational principle (4), provided that in any canonical coordinate frame the metric determinant $g(\xi)$ satisfies the condition

$$g(\xi) = h^{-2n}(\xi). \tag{9}$$

At first sight, this statement may sound quite strange, the replacement of the Hamiltonian $h(\xi)$ with the kinetic-energy-like term $\frac{1}{2}\mu g_{ij}\dot{\xi}^i\dot{\xi}^j$ making the original n degree of freedom Hamiltonian theory into a $2n$ degree of freedom Lagrangian theory. The variational principle (8) is in fact formally equivalent to that describing the free motion of a particle of mass μ (the "surface-scale" of the phase-space) on a metric manifold M (the phase-space endowed with the metric structure g_{ij}) coupled with a kind of universal magnetic field (the canonical two-form ω_{ij}) [10]. This magnetic analogy is actually quite useful in understanding the very small μ regime of the theory, and illustrates the mechanism producing the effective removal of the redundant degrees of freedom of the system. Before proving our statement let us therefore offer a few additional words about it.

A Magnetic Analogy

In order to visualize the problem in a very simple case let us consider a particle of mass m and charge e moving in a plane under the influence of a magnetic field of magnitude B normal to the plane. In our analogy the plane represents the phase space of a one-dimensional Hamiltonian system whereas the magnetic field its symplectic structure (see also, in a slightly different context [12, 13]). The limit of a very small mass corresponds to that of a very strong magnetic field or, equivalently, to that of a weakly-inhomogeneous magnetic field. The *phase-space motion of a dynamical system* will therefore be assimilated into the *adiabatic motion of a charged particle in an external magnetic field.* We can learn much about this subject in the literature. The problem, often referred to as *guiding center motion*, is in fact of primary interest in plasma physics and has been treated over the years by many authors from various points of view. An excellent review of the physical principles may be found in the book of T. G. Northrop [16]. In view of our interest in the canonical structure of the problem, we also refer to the works of C. S. Gardner [17], E. Witten [18], R. G. Littlejohn [19, 20] and P. Maraner [21].

As long as we consider a homogeneous magnetic field the particle follows a circular orbit of radius $r_B = \frac{mc}{eB}|\vec{v}|$ the center of which remains motionless. However, as soon as we introduce a weak inhomogeneity the center of the orbit starts moving, drifting slowly in the plane. The situation may be described inside the canonical formalism by introducing two pairs of canonical variables, the *adiabatic kinematical momenta* and the *adiabatic guiding center coordinates*. The former takes into account the fast rotation of the particle, whereas the latter the slow drift of the center of the orbit. We shall identify the guiding center motion with the phase space motion of the dynamical system and the fast rotation of the particle with the redundant degrees of freedom. The limit of a very small mass $m \to 0$, or, equivalently, of a very strong magnetic field $B \to \infty$, induces the circular orbit to collapse into a point so that only the guiding center motion remains detectable. The limit of small masses effectively removes the redundant degree of freedom from the theory simply because it lowers the degree of the classical equations

of motion.

Phase Space Motion in a Universal Magnetic Field

The standard analysis of guiding center motion deals with a Euclidean config-
uration space and an inhomogeneous magnetic field. For our consideration,
we are interested in a possibly more general situation in which the metric
structure may also vary from point to point. Fortunately, the qualitative pic-
ture of the system does not change since all that matters is the way in which
the magnetic field varies in the given geometry. To be more concrete let us
consider the Lagrangian $L(\xi, \dot{\xi}) = \frac{1}{2}\mu g_{ij}\dot{\xi}^i\dot{\xi}^j + \theta_i\dot{\xi}^i$. Introducing the canonical
momenta $p_i^\xi = \partial L/\partial \dot{\xi}^i$, we consider the corresponding Hamiltonian

$$ H(\xi, p^\xi) = \frac{1}{2\mu}g^{ij}(\xi)\left(p_i^\xi - \theta_i\right)\left(p_j^\xi - \theta_j\right), \tag{10} $$

g^{ij} denoting the inverse of the metric tensor. It is important not to confuse
this extended Hamiltonian theory with the original Hamiltonian theory. We
are no longer dealing with the phase space M, which now appears as the
configuration space of our extended system, but with its cotangent bundle
T^*M parametrized by the "positions" ξ and the "momenta" p^ξ [1, 2]. In order
to avoid any confusion between the original n degrees of freedom Hamiltonian
system and our extended $2n$ degrees of freedom Hamiltonian system we shall
denote the Poisson brackets on T^*M by $\{\!\{F, G\}\!\} = \frac{\partial F}{\partial \xi^i}\frac{\partial G}{\partial p_i^\xi} - \frac{\partial F}{\partial \xi^i}\frac{\partial G}{\partial p_i^\xi}$.

Kinematical Momenta and Guiding Center Coordinates

Let us proceed by observing that the form of the Hamiltonian (10) may be
simplified considerably by first replacing the canonical momenta p_i^ξ with the
gauge covariant *kinematical momenta*

$$ \Pi_i = \frac{1}{\mu^{1/2}}\left(p_i^\xi - \theta_i\right), \tag{11} $$

$i = 1, ..., 2n$. Up to a scale factor Π_ν and $\Pi_{n+\nu}$, $\nu = 1, ..., n$, behave as
conjugate variables so that (10) becomes the Hamiltonian of an n-dimensional
harmonic oscillator with masses and frequencies depending on ξ. Since the

Poisson brackets between the Π_is and the ξ^is are in general different from zero, $\{\!\{\xi, \Pi\}\!\} \neq 0$, we are led to further adapt our phase space variables by introducing the *guiding center coordinates*

$$X^i = \xi^i + \mu^{1/2}\bar{\omega}^{ij}\Pi_j, \tag{12}$$

$i = 1, ..., 2n$. In our magnetic analogy the Π_is describe the fast rotation of the particle around the guiding center, whereas the X^is take into account the slow drift of the center of the orbit. The new set of variables fulfills the Poisson bracket relations

$$\begin{aligned}
\{\!\{\Pi_i, \Pi_j\}\!\} &= \mu^{-1}\,\omega_{ij}, \\
\{\!\{\Pi_i, X^j\}\!\} &= 0, \\
\{\!\{X^i, X^j\}\!\} &= \bar{\omega}^{ji},
\end{aligned} \tag{13}$$

so that the guiding center coordinates and kinematical momenta may be recognized as a new set of canonical variables (cf. expressions (3) and (5)). The presence of the scale factor μ^{-1} allows us to identify the Π_is and the X^is as describing respectively *fast* and *slow* degrees of freedom of the system [21]. Rewriting the Hamiltonian (10) in terms of the new variables and expanding in the small parameter $\mu^{1/2}$ we find that

$$\mathrm{H}(X, \Pi) = \frac{1}{2}g^{ij}(X)\Pi_i\Pi_j + \mathcal{O}(\mu^{1/2}). \tag{14}$$

The relevant term of the expansion looks again like an n-dimensional harmonic oscillator in the fast variables Π_is the parameters depending this time only on the slow variables X^is.

A second canonical transformation

The dynamics of fast and slow degrees of freedom may be separated, up to terms of order $\mu^{1/2}$, by performing a second canonical transformation. For this task we decompose the inverse metric $g^{ij}(X)$ as $g^{ij}(X) = g^{-1/2n}(X)\gamma^{ij}(X)$, $g(X)$ being the determinant of the metric and $\gamma^{ij}(X)$ a point-dependent matrix with determinant one. We further represent $\gamma^{ij}(X)$ by means of $2n$-beins

as $\gamma^{ij}(X) = \delta^{kl}t_k^i(X)t_l^j(X)$. Making use of the condition (9) the inverse metric may thus be written as

$$g^{ij}(X) = h(X) \ \delta^{kl}t_k^i(X)t_l^j(X), \tag{15}$$

where $h(X)$ is the Hamiltonian of our original system. Denoting by $\tau_j^i(X)$ the logarithm of the $2n$-bein $t_j^i(X)$, $\tau(X) = \ln t(X)$, we perform a canonical transformation generated by the function $\Lambda(X, \Pi) = \frac{1}{2}\tau_k^i(X)\bar{\omega}^{kj}\Pi_i\Pi_j$, the infinitesimal parameter being identified with μ. The variables produced by the transformation again fulfill the Poisson brackets relations (13) so that the new phase space coordinates are again separated into the two canonical subsets $\{X'\}$ and $\{\Pi'\}$, X'^ν being conjugate to $X'^{n+\nu}$ and Π'_ν to $\Pi'_{n+\nu}$, $\nu = 1, ..., n$. Up to terms of order μ it follows that

$$\begin{cases} X'^i &= X^i + \mathcal{O}(\mu) \\ \Pi'_i &= t_i^k(X)\Pi_k + \mathcal{O}(\mu) \end{cases}. \tag{16}$$

In terms of the new variables the Hamiltonian (10) separates into a product of a function of the X'^is times a function of the Π'_is

$$\mathrm{H}(X', \Pi') = h(X') \ J + \mathcal{O}(\mu^{1/2}), \tag{17}$$

$J = \frac{1}{2}\sum_i \Pi_i'^2$ representing the Hamiltonian of an n-dimensional harmonic oscillator.

(Effective) Hamiltonian Dynamics

Disregarding higher order terms, the mechanics of the X'^is completely separates from that of the Π'_is. The X'^is describe an n degree of freedom Hamiltonian system the phase space of which may be identified with M and whose dynamics is characterized by the Hamiltonian h, namely our original Hamiltonian system. The Π'_is, on the other hand, describe an n dimensional harmonic oscillator performing, for fixed energy, vibrations of amplitude μ and frequency μ^{-1}. By decreasing the value of μ the orbits of our extended system collapse therefore into ones of the original Hamiltonian system, the presence of the redundant variables Π' becoming increasingly irrelevant.

We would like to stress that we are not considering the limit procedure $\mu \to 0$ in a rigorous mathematical sense. In our present viewpoint $\mu^{1/2}$ represents a very small but *finite* parameter, capable of a concrete physical interpretation. It represents the phase-space-length-scale over which the universal magnetic field represented by the symplectic two-form ω_{ij} may be considered as homogeneous. On the other hand, it is the inhomogeneities on larger scales that produce dynamics.

2.3   The Harmonic Oscillator Problem as a Simple Example of the Method

For the sake of completeness let us write down explicitly the geometrical equations driving our dynamical theory. Consider an n-degree of freedom system described by the (positive definite) Hamiltonian $h(\xi)$. On the phase space M we introduce the metric

$$g_{ij}(\xi) = \frac{1}{h(\xi)} \gamma_{ij}(\xi),$$ (18)

γ_{ij} being a point dependent $2n$ by $2n$ matrix with determinant one. The choice of γ_{ij} is obviously related to the topological features of the phase space. As long as we are interested in the adiabatic regime its explicit form does not play any role and its choice is purely a matter of convenience. For a flat topology we may choose the Kronecker delta, $\gamma_{ij} = \delta_{ij}$, while non trivial topologies generally require more complicated expressions. The equations of motion follow from the Lagrangian $L(\xi, \dot{\xi})$ as

$$\ddot{\xi}^k + \Gamma_{ij}^k \dot{\xi}^i \dot{\xi}^j = \frac{1}{\mu} g^{ki} \omega_{ij} \dot{\xi}^j$$ (19)

$k = 1, ..., 2n$ and $\Gamma_{ij}^k = g^{kl}(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij})/2$ denoting the Christoffel symbols relative to the connection induced on M by g_{ij}. Aside from the magnetic term on the right hand side, these correspond to the geodesic equations for a free motion on M. Nevertheless, it has to be pointed out that the presence of the Lorentz like term $\frac{1}{\mu} g^{ki} \omega_{ij} \dot{\xi}^j$ can drastically modify the behaviour of the system, even for large values of μ. By decreasing the value of μ further,

the trajectories of our system tightly wrap around the ones of the original Hamiltonian system, becoming physically indistinguishable from these for very small values of μ. In order to illustrate these features in a concrete example let us discuss in some detail the harmonic oscillator problem.

The harmonic oscillator

Consider a one-dimensional harmonic oscillator described by the Hamiltonian $h(p, q) = \frac{1}{2}(p^2 + q^2)$. The topology of the phase space being trivial we choose the metric tensor $g_{ij}(p, q) = 2\delta_{ij}/(p^2 + q^2)$. This make the phase plane into an infinite cylinder, the extremities of which have to be identified with the inaccessible point zero[2] and the point at infinity. To make this explicit we introduce non-canonical cylindrical coordinates (ρ, ϕ) related to (p, q) by the transformation $q = \mu^{1/2} e^{-\rho} \sin\phi$, $p = \mu^{1/2} e^{-\rho} \cos\phi$. Choosing the symmetric gauge for the canonical one-form θ_i, the phase space Lagrangian of the system reads

$$L(\rho, \phi, \dot{\rho}, \dot{\phi}) = \mu \left(\dot{\rho}^2 + \dot{\phi}^2 \right) + \frac{\mu}{2} e^{-2\rho} \dot{\phi} \tag{20}$$

making clear the formal analogy of our system with a particle moving on a cylinder in an orthogonal magnetic field of magnitude $B(\rho) \simeq -e^{-2\rho}$. The presence of the magnetic term makes the region $\rho = -\infty$ inaccessible, dramatically modifying the free trajectories of the system. The geodesics on the cylinder are in fact represented by circles of constant ρ and helices escaping toward both extremities with constant velocity. In the phase plane picture of the cylinder these trajectories are represented respectively by circles, $r(t) \equiv \sqrt{p^2(t) + q^2(t)} = const$, and by spirals collapsing onto the origin, $r(t) \sim r_0 e^{-kt}$, or escaping to infinity, $r(t) \sim r_0 e^{kt}$. For every value of μ the magnetic force removes the trajectories escaping to infinity by confining the motion to a neighbourhood of the origin.

---

[2]The origin may be made into an accessible point for the system by adding a positive constant to the Hamiltonian and hence to the conformal factor of the metric. This modifies the geometry of the phase plane (it is no longer flat) but not the adiabatic regime of the dynamics. Moreover, our description of dynamics is in some sense fuzzy. We consider
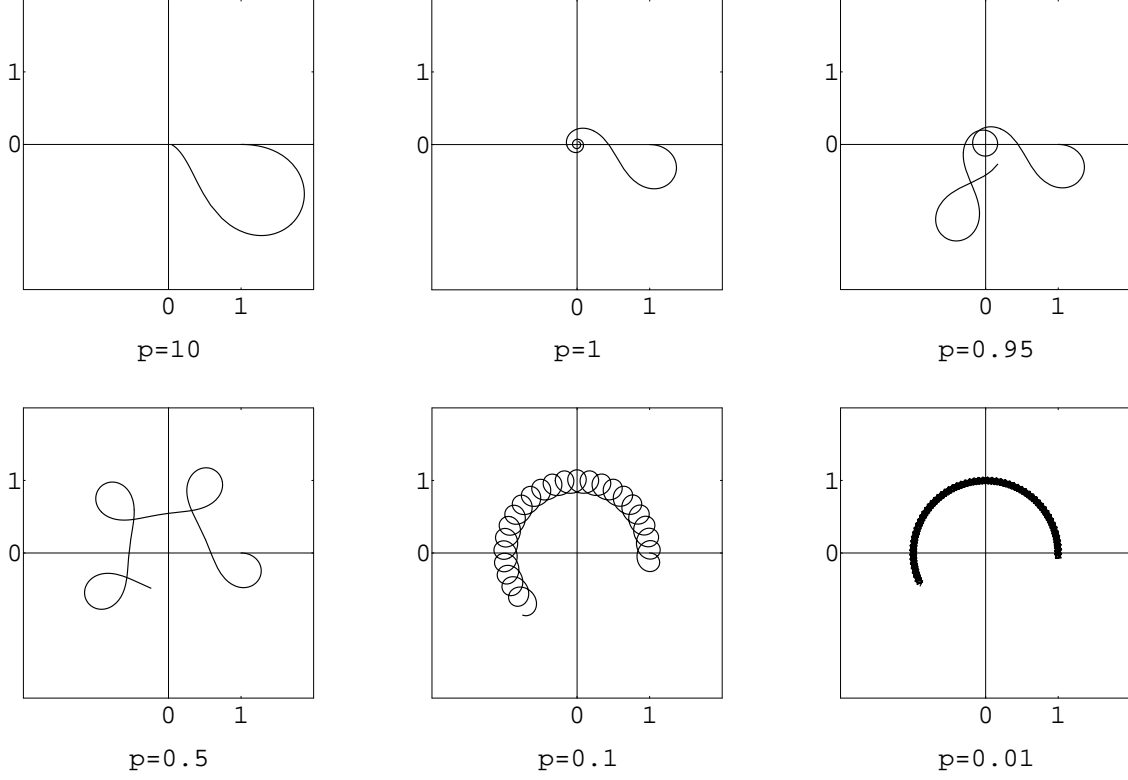
Figure 1: Phase space motion of the representative point of the system for different values of the parameter $\mathbf{p} = \frac{\mu \mathcal{E}}{l^2}$. The system is initially in the point $(1, 0)$. The "energy" $\mathcal{E}$ and the "angular momentum" l are fixed to the values 1 and $\frac{1}{4}$ respectively.

In order to proceed to the solution of the dynamical problem we consider the two integrals of motion of the system, the analogues of angular momentum and energy for the equivalent particle moving on the cylinder,

$$\mu \, \dot{\phi} + \frac{1}{4} \, r^2 = l, \tag{21}$$

$$\mu \, \frac{\dot{r}^2}{r^2} + \mu \, \dot{\phi}^2 = \mathcal{E}. \tag{22}$$

phase space points infinitesimally close to each other as indistinguishable physical states so that the loss of a single phase space point does not constitute a serious problem.

By eliminating $\dot{\phi}$ in (22) by means of (21) (we suppose $l \neq 0$) we see that the motion in the ρ direction takes place in a Morse potential. Introducing the variable $\zeta = \frac{\mu}{4l}e^{-2\rho} - 1$ the quadrature of the problem is then reduced to the evaluation of the integral

$$t - t_0 = \pm \frac{\mu}{2l} \int_{\zeta_0}^{\zeta} \frac{d\zeta}{(\zeta + 1)\sqrt{\frac{\mu\mathcal{E}}{l^2} - \zeta^2}}. \tag{23}$$

This yields

$$r^2(t) = \begin{cases} \dfrac{2l\,(\mathbf{p} - 1)}{e^{\pm 2l\sqrt{\mathbf{p}-1}(t-t_0)/\mu} + \mathbf{p}\,e^{\mp 2l\sqrt{\mathbf{p}-1}(t-t_0)/\mu} - 2} & \text{for } \mathbf{p} > 1 \\[4mm] \dfrac{8l}{1 + 4l^2(t - t_0)^2/\mu^2} & \text{for } \mathbf{p} = 1 \\[4mm] \dfrac{4l\,(1 - \mathbf{p})}{1 \pm \mathbf{p}^{1/2}\sin\left[2l(1 - \mathbf{p})^{1/2}(t - t_0)/\mu\right]} & \text{for } \mathbf{p} < 1 \end{cases} \tag{24}$$

The behaviour of the system depends on the parameter $\mathbf{p} = \mu\mathcal{E}/l^2$, its value being greater, equal or lesser than one producing three different dynamical regimes. The trajectories with $\mathbf{p} > 1$ correspond to unbound states of the Morse potential. In the phase plane picture of the system the representative point falls onto the origin with the exponential law $r(t) \sim e^{-t}$ (Fig.1, $\mathbf{p} = 10$). For $\mathbf{p} = 1$ the "energy" of the system equals the asymptotic limit of the Morse potential so that the motion is again unbounded. The phase space trajectories again fall onto the origin but with the power law $r(t) \sim 1/t$ (Fig.1, $\mathbf{p} = 1$). Finally, for $\mathbf{p} < 1$, we obtain the bound states of the Morse potential. The representative point of the system neither falls onto the phase plane origin nor escapes to infinity.

Whereas for $\mathbf{p} \geq 1$ the trajectories of the system share a quite simple form, for values of $\mathbf{p}$ very close to one from below the representative point of the system makes rather unusual curves on the phase plane trying to fall onto the origin but returning over and over to a neighbourhood of the starting point (Fig.1, $\mathbf{p} = 0.95$). For fixed values of $\mathcal{E}$ and l the adiabatic limit of the theory is reached for very small values of μ. By decreasing μ, in

17

fact, the oscillations of $r(t)$ and also of $\phi(t)$ are strongly damped so that for very small μ the system follows a thick spiral of very small radius wrapping around a circle, that is a phase space trajectory of the harmonic oscillator (Fig.1, $\mathtt{p} = 0.5, \mathtt{p} = 0.1, \mathtt{p} = 0.01$). This is exactly the adiabatic behaviour we have predicted in general terms.

3 'Free' Quantum Dynamics

3.1 The Geometrical Background of Quantization

Some thirty years ago the problem of quantizing a general Hamiltonian system has been seriously faced for the first time in the so called Geometric Quantization scheme of B. Kostant, A. Kirillov and J. M. Souriau [5, 6, 7, 4]. Geometric Quantization should not be considered by the same standard as the several physics-generated quantization procedures that have been proposed over the years. Rather, it should be regarded as an analysis of the various structures needed for the quantization of a classical system, providing the proper mathematical background and the right mathematical tools necessary to analyze the issues surrounding quantization. On the other hand, Geometric Quantization lacks physical intuition and, as a matter of fact, it has succeeded more in pointing out the formal difficulties involved in the quantization procedure than in providing their solution. Though in what follows we will make only an implicit use of the abstract tools introduced by Geometric Quantization, this language exactly corresponds to the one to be employed in the description of a charged quantum particle in a non-trivial topology, that is, taking into account the magnetic analogy we discussed in the previous sections, in our dynamical theory (see also [10]). We find it worthwhile, therefore, to briefly recall the salient features of the construction.

In discussing a field theory like quantum mechanics in a non-trivial topological context it is necessary to pay attention in treating global features [22]. Although everything should make sense globally not every object appearing in the theory is capable of a global definition. As a relevant example, once a phase space with a non-trivial topology is considered the canonical one-form

18

θ_i is only locally defined (like the vector potential of an Aharonov-Bohm magnetic field). This quantity, appearing directly in the Hamiltonian action, forces the wave functions of the system to share the same undesirable feature. The problem, nevertheless, does not concern the theory as a whole but only its local representation and an appropriate language to deal with the situation has to be introduced. This is fiber bundle theory [4]. We have in some sense to be content with a piecewise representation of the theory making sure that when moving from one local representation to another, everything makes sense globally. In constructing a coordinate-free quantization procedure, therefore, we have to take the symplectic two-form as the *curvature form* of an appropriate *line bundle* over the phase space M. The (only locally defined) canonical one-form appears then as the corresponding *connection form* while the wave functions of the system acquire a global meaning as *sections* of the line bundle [5]. The practical results of this elegant construction [5]—which is the only way to give a global meaning to the world "quantization"—is that the so constructed Hilbert space appears to be too large and some additional structure must be picked out on the phase space M in order to reduce its dimension. This brings us back to the introductory section and to the discussion concerning real/complex polarizations and phase space metric structures. For details we refer to the original works quoted above. An interesting approach, similar in many respects to that of polarization, has also being recently developed by E. Gozzi [24].

3.2 Quantizing 'Free' Dynamics

The task of giving a fully geometrical picture of the dynamical mechanism leading to the set of physical states for a quantum system is the main motivation which has brought us to a description of standard Hamiltonian mechanics as the adiabatic limit of a fully geometrical phase-space-theory. Once classical dynamics is re-expressed in terms of the variational principle (8), the task of quantizing the classical system is reduced to standard procedures.

Path Integral Approach

The basic features of this approach may be seen immediately by writing down the formal phase space expression of the propagator

$$K(\xi'', t''; \xi', t') = \int e^{i \int \left(\frac{1}{2} g_{ij} \dot{\xi}^i \dot{\xi}^j + \frac{1}{\mu} \theta_i \dot{\xi}^i \right) dt} \mathcal{D}\xi; \tag{25}$$

the presence of the kinetic-energy-like term $\frac{1}{2}\mu g_{ij}\dot{\xi}^i\dot{\xi}^j$ in the phase space action enables one to give a precise—although not unique since ordering ambiguities are still present—meaning to this expression by means of an imaginary time continuation and a Wiener measure on M, exactly as in Klauder's quantization scheme [9]. Nevertheless, in the present context we need not perform any limiting procedure to remove any regulator, the phase space metric playing now an essential dynamical role in the theory.

Hamiltonian Approach

An alternative way to look at the standard nature of quantization in our scheme is to think of the magnetic analogy. The problem is equivalent to that of quantizing a particle moving on a metric manifold M in the universal magnetic field ω_{ij}. As sketched in the previous section, in discussing the motion of a charged quantum particle in an external magnetic field in a non-trivial topology, it is necessary to treat global properties very carefully. On the other hand, the problem is a fairly standard one. From the work of T. T. Wu and C. N. Yang on the geometrical setting of Dirac's monopole theory [22] we learn that the magnetic field and vector potential (the canonical two-form and one-form, in our context) have to be considered respectively as the curvature two-form and the connection one-form of an appropriate line bundle over the configuration space (the phase space M, in our context), while the states of the system have to be identified with sections of this line bundle (see also [12, 23]). The whole apparatus of geometric quantization reappears therefore in a very natural and necessary manner. The Hamiltonian operator associated to the propagator (25) is also capable of a *global* definition in terms of the Laplacian over the considered line bundle and, eventually, invariant counterterms constructed by means of the phase space metric and

symplectic structures. In any coordinate frame the quantum Hamiltonian will appear as

$$\mathrm{H} = \frac{1}{2g^{1/2}(\xi)} \, \Pi_i \, g^{ij}(\xi) \, g^{1/2}(\xi) \Pi_j + \mu \, \mathcal{I}_1 + \mu^2 \, \mathcal{I}_2 + ..., \tag{26}$$

where we have introduced the kinematical momenta $\Pi_i = -i\mu^{1/2}\partial_i - \theta_i/\mu^{1/2}$ and $\mathcal{I}_1, \mathcal{I}_2$, etc. , are "optional" invariants whose presence reflects the ordering ambiguities inherent in the quantization procedure. As an example $\mathcal{I}_1$ may contain a term proportional to the phase space scalar curvature R [25], but also other invariants with the right dimension constructed from the covariant derivatives of ω_{ij} are possible. These invariants produce $\mathcal{O}(\hbar^2)$ corrections to the spectrum of the system, effects which are generally small. For the moment we do not care to make any particular choice of them; a quite natural choice will appear later.

We observe that the Hamiltonian (26) acts on wave functions depending on all the phase space coordinates $\xi = (q, p)$ so that at first sight it may appear that our theory shares the same difficulties as Kostant's prequantization scheme. However, an analysis along the same lines as that in section 2.2 indicates that the system provides, *by itself*, the means to remove the unphysical degrees of freedom, the intuitive picture to keep in mind being always that of Fig.1.

Kinematical Momenta and Guiding Center Operators

In close analogy with our discussion of the classical theory we introduce, besides the kinematical momenta Π_i, the guiding center operators $X^i = \xi^i + \mu^{1/2}\bar\omega^{ij}\Pi_j$ obtaining a new set of observables. In any canonical coordinate frame the local representation of the X's and Π's as differential operators satisfy the canonical commutation relations

$$\begin{aligned}
\left[\Pi_i, \Pi_j\right] &= i\omega_{ij}, \\
\left[\Pi_i, X^j\right] &= 0, \\
\left[X^i, X^j\right] &= i\mu\,\bar\omega^{ji}.
\end{aligned} \tag{27}$$

21

It is nevertheless important to stress that, unless the topology of the phase space M is trivial, these commutation relations hold only *locally*! That is to say the X's and Π's do not constitute in general a global representation of the Heisenberg algebra. On the other hand, we are not trying to construct a quantization procedure in the standard sense, namely looking for a correspondence between the algebra of classical and quantum observables; all that we are looking for is a global definition of the dynamics of the system and (26) is in fact (a local representation of) a globally well defined object.

The Adiabatic Expansion

Replacing ξ^i with $X^i - \mu^{1/2}\bar{\omega}^{ij}\Pi_j$ in (26) and expanding in power of $\mu^{1/2}$ we obtain the quantum analog of equation (14). As in the classical case the X and Π degrees of freedom may be separated up to terms of order $\mu^{1/2}$ by performing a unitary transformation generated by the Hermitian operator $\Lambda(X,\Pi) = \frac{1}{2}\tau_k^i\bar{\omega}^{kj}\{\Pi_i,\Pi_j\}$ ($\{\,,\,\}$ denotes anticommutators here). By successive suitable unitary transformations it is also possible to make all the half-integer order terms of the perturbative expansion vanish identically, while the integer order terms may be written as geometric invariants evaluated in the X's times powers of the harmonic oscillator Hamiltonian $J = \frac{1}{2}\sum_i \Pi_i^2$ constructed by means of the Π's (the method to be used is a straightforward generalization of a well-known technique of perturbation theory in classical mechanics and has been developed in [21]).

Denoting again by X^i and Π_i the new "canonical" operators—fulfilling (27) in every canonical coordinate frame—the quantum Hamiltonian describing our system takes on the form

$$\mathrm{H} = h(X)\,J + \mathcal{O}(\mu). \tag{28}$$

The original Hamiltonian $h(\xi)$ is here evaluated in the set of non-commuting operators X, an operation involving ordering ambiguities. It is on the other hand immediately realized that a different choice of ordering modifies only the higher order terms of the expansion, terms which are already not uniquely defined in virtue of the freedom in the choice of the invariants $\mathcal{I}_1$, $\mathcal{I}_2$, etc. .

(Effective) Quantum Dynamics

The dynamics of the $2n$ canonically conjugate *slow* variables X's separates from that of the *fast* Π's. The energy necessary to induce a transition in the spectrum of the fast variables being much greater than the energy scale involved in the slow motion, the system may be considered as frozen in one of the J eigenstates and the effective dynamics pertains only to the evolution of the slow variables. In other words, the system, by itself, effectively removes dynamically the redundant (physically unobservable) degrees of freedom. The higher order terms of the perturbative expansion being operators depending on the variables X's—commuting to i times the adiabatic parameter μ—contribute to the spectrum of the system with corrections of order higher than μ^2. Moreover, once the system is frozen in an eigenstate of J, presumably its ground state, it is possible in principle to perform a choice of the invariants $\mathcal{I}_1$, $\mathcal{I}_2$, etc. , in such a way that the whole adiabatic expansion except for the zero order term identically vanishes for that state. The scheme therefore allows a reproduction of all the ordering prescriptions and even something more.

In concluding this section let us observe what the reader probably already suspects. A rapid look at the commutation relations (27), the Hamiltonian (28) and even the propagator (25), makes clear that the adiabatic parameter μ should be identified with Planck's constant

$$\mu \equiv \hbar. \tag{29}$$

Hereafter, we shall assume this equality. In our picture, therefore, Planck's constant assume an intuitive geometrical meaning: $\hbar^{1/2}$ is the natural phase-space-length-scale measuring the inhomogeneity of the universal magnetic field ω_{ij} in the metric g_{ij}.

3.3   One Degree of Freedom Systems

In order to illustrate in more detail the method and to compare it with standard quantization procedures in a trivial topological context we specialize to one degree of freedom systems. The phase space to be considered is then

23

represented by a two-dimensional surface M while—as in the general case—quantum kinematics and dynamics are completely characterized by (25)/(26) once symplectic and metric structures are assigned. In every canonical coordinate frame $\xi = (q, p)$

$$\omega_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad g_{ij} = \frac{1}{h(\xi)} \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix}, \tag{30}$$

where the metric has again been factored into the product of a function times a point dependent matrix with determinant one as in (18). Let us observe that in the case of one degree of freedom systems this decomposition has a special covariant character. The inverse conformal factor $h(\xi)$ corresponds in fact to the norm of the symplectic two-form ω_{ij}, $h(\xi) = \sqrt{\omega_{ij}\omega^{ij}/2}$. $h(\xi)$ transforms therefore as a scalar while γ_{ij} as a symmetric two-tensor.

Suppose now that the topology of the surface M is compatible with a flat geometry. This is the case, as an example, of the harmonic oscillator discussed in section 2.3 and of most dynamical system usually considered in textbooks. Without affecting the adiabatic regime of the theory—that is dynamics—it is then possible to choose the tensor γ_{ij} in such a way that g_{ij} is flat. Performing this choice eliminates geometrical complications, the problem resulting being equivalent (up to boundary conditions) to the motion of a charged spinless particle in a plane under the influence of a perpendicular inhomogeneous magnetic field. To make this explicit we introduce Cartesian (non-canonical) coordinates $\bar{\xi} = \bar{\xi}(\xi)$. The metric tensor then becomes a Kronecker delta while it follows that the symplectic two-form is simply multiplied by its norm,

$$\bar{\omega}_{ij} = h(\bar{\xi}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \bar{g}_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{31}$$

The bar indicates that the tensors are to be evaluated in the new coordinates while $h(\bar{\xi})$ should be interpreted as $h(\xi(\bar{\xi}))$. In the Cartesian background the Hamiltonian (26) becomes

$$\mathrm{H} = \frac{1}{2}\delta^{ij}\bar{\Pi}_i\bar{\Pi}_j + \hbar\,\mathcal{I}_1 + \hbar^2\,\mathcal{I}_2 + \dots, \tag{32}$$

$\bar{\Pi}_i = \partial \xi^k / \partial \bar{\xi}^i \Pi_k$ denoting the Cartesian kinematical momenta, and the invariants $\mathcal{I}_1$, $\mathcal{I}_2$, etc. , are evaluated in $\bar{\xi}$. Obviously $\bar{\Pi}_1$ and $\bar{\Pi}_2$ are no longer conjugate variables. The new set of operators $\bar{\Pi}$'s and $\bar{\xi}$'s in fact fulfill the commutation relations

$$\begin{aligned}
\left[\bar{\Pi}_i, \bar{\Pi}_j\right] &= i\, h(\bar{\xi})\, \omega_{ij}, \\
\left[\bar{\Pi}_i, \bar{\xi}^j\right] &= -i\, \hbar^{1/2}\, \delta_i^j, \\
\left[\bar{\xi}^i, \bar{\xi}^j\right] &= 0.
\end{aligned} \tag{33}$$

The Hamiltonian (32) together with (33) makes explicit the analogy of the problem with the motion of a quantum charged particle in a plane under the influence of the magnetic field $B(\bar{\xi}) = h(\bar{\xi})$ [21]. The adiabatic regime of this theory has been recently investigated by one of us, P. Maraner, obtaining the explicit expression of the first few terms of the adiabatic expansion. Introducing in a suitable way *adiabatic kinematical momenta* and *adiabatic guiding center operators* the Hamiltonian (32) becomes (see [21] for details)

$$\mathrm{H} = h\,\bar{J} + \frac{\hbar}{4}\left[\frac{\triangle h}{h} - 3\frac{|\nabla h|^2}{h^2}\right]\,\bar{J}^2 + \frac{\hbar}{16}\left[\frac{\triangle h}{h} - \frac{|\nabla h|^2}{h^2}\right] + \hbar\mathcal{I}_1 + \mathcal{O}(\hbar^2), \tag{34}$$

where $\bar{J}$ represents the harmonic oscillator Hamiltonian constructed by means of the adiabatic kinematical momenta and all the scalars are evaluated in the adiabatic guiding center operators. Freezing the fast variable of the system in its ground state and transforming back to the original canonical frame the effective Hamiltonian $h^{(eff)}$ describing the slow motion is obtained as

$$h^{(eff)} = \frac{1}{2}h(X) + \hbar\left[\frac{1}{8}\frac{\triangle h}{h}(X) - \frac{1}{4}\frac{|\nabla h|^2}{h^2}(X) + \mathcal{I}_1(X)\right] + \mathcal{O}(\hbar^2), \tag{35}$$

$X^i = \xi^i + \hbar^{1/2}\bar{\omega}^{ij}\Pi_j$, $i = 1, 2$, again denoting the guiding center operators introduced in the previous section. For any arbitrarily assigned ordering prescription, the choice (compare also [11])

$$\mathcal{I}_1 = \frac{1}{4}\frac{|\nabla h|^2}{h^2} - \frac{1}{8}\frac{\triangle h}{h} \tag{36}$$

makes our quantization scheme reproduce the standard one up to terms of order $\hbar^3$. It is also possible, at least in principle, to proceed by choosing all the remaining invariants $\mathcal{I}_2$, $\mathcal{I}_3$, etc. , is such a way that the whole perturbative expansion except the zero order term vanishes identically. Aside from an inessential multiplicative factor $1/2$ the *(effective) quantum dynamics* of the system is described by

$$h^{(eff)} = h(Q, P), \tag{37}$$

$Q \equiv X^1$ and $P \equiv X^2$ being a pair of conjugate operators, $[Q, P] = i\hbar$, and where an ordering choice has been performed.

4 Discussion and Speculations

Starting from the generally well accepted opinion that quantization involves picking out some *additional structure* on the phase space M of a classical system we have speculated on the possibility of describing classical as well as quantum dynamics by means of a *phase space metric structure*. This produces an $\mathcal{O}(\hbar)$ modification of the classical equations of motion reducing at the same time the problem of quantizing an arbitrary Hamiltonian system to standard procedures. Our analysis nevertheless appears as unconventional. We do **not** insist, in fact, on a unique correspondence between classical and quantum states, observables and evolution equations. All that we care about is giving a global definition of quantum dynamics in the Hilbert space of square integrable functions on the classical phase space M (see (25), (26)). The system then provides *by itself* the dynamical selection of the subspace of physical states. Moreover, our scheme does not yield a unique answer to quantization. Questions connected with ordering are still present in the theory. On the other hand, as long as various physical situations potentially involve different orderings, it is our opinion that a sensible quantization scheme should give not *one quantization* but *"all" possible quantizations* of any given classical system.

In our view, dynamics appears in a very interesting way as a purely geometrical effect, in formal analogy with the guiding center motion of a charged

26

particle in a curved background (*the additional metric structure*) under the influence of a universal magnetic field (*the classical symplectic structure*). In the present paper we have restricted our attention to non-singular symplectic structures, that is to unconstrained systems. Nevertheless, there is no problem, at least in principle, in extending our discussion to singular symplectic structures since the dynamics is supported by the metric. As a very simple but nontrivial example we may consider motion in a three dimensional phase space. The symplectic structure is then singular and yet we can still picture the behaviour of the system by means of the motion of a particle in an ordinary three-dimensional space under the influence of an arbitrary magnetic field (compare with section 3.3). The resulting adiabatic picture [16] is that of a system moving freely along the field lines (the "unphysical" part of dynamics) while rapidly rotating around its guiding center (the unobservable degree of freedom) and drifting in the directions normal to the field (the "physical" part of dynamics). The principal obstacle in extracting an explicit form for the Hamiltonian describing the effective guiding center motion, namely the physically relevant part of dynamics, is deeply connected with the problem of finding a local Darboux coordinate frame in which the magnetic field reduces to the canonical form [17]

$$\omega_{ij} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Succeeding in this task is on the other hand equivalent to the so called *abelianization of the constraints* representing a complete separation of the physical and unphysical degrees of freedom, and which leads directly to the solution of the problem. What appears interesting from our point of view is that, in the study of guiding center motion, techniques have been developed to describe the adiabatic regime of the dynamics without directly appealing to the explicit form of the Darboux transformation [26]. Our scheme appears therefore as promising in dealing with the quantization of *constrained systems*.

From a more speculative viewpoint other interesting questions may be addressed:

27

We may wonder, as an example, if the "unobservable" degrees of freedom represented by the fast rotation of the system around its guiding center are capable of a physical interpretation (that is, if they are observable after all). A reasonable guess would be that the $SU(n)$ hidden symmetry of our dynamics may accomodate the *spin* degrees of freedom of a quantum system. To clarify this point one needs to study the response of the system to an external magnetic field, which may be incorporated into the theory as a local modification of the symplectic structure.

More ambitiously, one may speculate on the possibility that the $\mathcal{O}(\hbar)$ modification of classical mechanics presented in Eq.8 is in some way related to quantum mechanics itself—without going through quantization—as the fuzzy trajectories of Fig.1 may suggest. Nevertheless, even in the solution of the simple harmonic oscillator problem there is no trace of quantization and every attempt at constructing a statistical theory based on a deterministic background must deal with Bell's theorem.

Finally, one may wonder about the possibility of giving a dynamical role to our metric, relating phase-space-geometry to the phase-space-matter-distribution in a way reminding one of general relativity.

At the moment, however, these points go well beyond our original purpose.

Acknoledgments

References

[1] R. Abraham and J. E. Marsden, *Foundations of Mechanics* (Benjamin, London, 1978)

[2] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1978)

[3] P. A. M. Dirac, *Quantum Mechanics* (Oxford University Press, London, 1958)

[4] N. Woodhouse, *Geometric Quantization* (Clarendon Press, Oxford, 1980)

[5] B. Kostant, in *Quantization and Unitary Representations, Lecture Notes in Mathematics*, C. C. Taum ed. (Springer, New York, 1970)

[6] A. Kirillov, *Eléments de la Théorie des Représentations* (MIR, Moscou, 1974)

[7] J. M. Souriau, *Structure des Systèmes Dynamiques* (Dunod, Paris, 1970)

[8] E. Onofri, *J. Math. Phys.* **17**, 401 (1976)

[9] J. R. Klauder, *Ann. Phys.* **188**, 120 (1988)

[10] J. R. Klauder and E. Onofri, *Int. J. Mod. Phys.* **A4**, 3930 (1989)

[11] P. Maraner, E. Onofri and G.P. Tecchiolli, *Mod. Phys. Lett.* **A7**, 1377 (1992)

[12] P. Maraner, *Mod. Phys. Lett.* **A7**, 2555 (1992)

[13] R. Alicki, J. R. Klauder and J. Lewandowski, *Phys. Rev.* **A48**, 2538 (1993)

[14] R. Alicki and J. R. Klauder, *J. Phys. A: Math. Gen.* **29**, 2475 (1996)

[15] J. R. Klauder, *"Geometric Quantization from a Coherent State Viewpoint"* (to appear in the proceedings of the *"XIVth Workshop on Geometric Methods in Physics"*, Bialowieża, Poland, July 9-15 1995)

[16] T. G. Northrop, *The Adiabatic Motion of Charged Particles* (Intersience, New York, 1963)

[17] C. S. Gardner, *Phys. Rev.* **115**, 791 (1959)

[18] E. Witten, *Ann. Phys.* **120**, 72 (1979)

[19] R. G. Littlejohn, *J. Math. Phys.* **12**, 2445 (1979)

[20] R. G. Littlejohn, in *Contemporary Mathematics* Vol.**28**, 151, J. E. Marsden ed. , (American Mathematical Society, 1984)

[21] P. Maraner, *J. Phys. A: Math. Gen.* **29**, 2199 (1996)

[22] T. T. Wu and C. N. Yang, *Phys. Rev.* **D12**, 3845 (1975); T. T. Wu and C. N. Yang, *Nuc. Phys.* **B107**, 365 (1976); C. N. Yang, in *Understanding The Fundamental Constituents of Matter*, A. Zichichi ed. , (Plenum, New York, 1978)

[23] M. Daniel and C. M. Viallet, *Rev. Mod. Phys.* **52**, 175 (1980)

[24] E. Gozzi, *Phys. Lett.* **A202**, 330 (1995)

[25] B. De Witt, *Phys. Rev.* **85**, 635 (1952)

[26] R. G. Littlejohn, *Phys. Fluids.* **29**, (1981)